Determinacy, measure, toasts, and the shift graph

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McGill DDC Seminar

On Borel combinatorics

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Can talk about:

Borel graphs: G is a Borel graph is G is Borel a subset of $V(G) \times V(G)$.

Borel chromatic numbers: minimal n for which G has a Borel n-coloring. Notation: $\chi_B(G)$.

Borel homomorphisms: if G, H are Borel graphs, a Borel homomorphism is a Borel map $f : V(G) \to V(H)$ that takes edges to edges. Notation: $G \leq_B H$.

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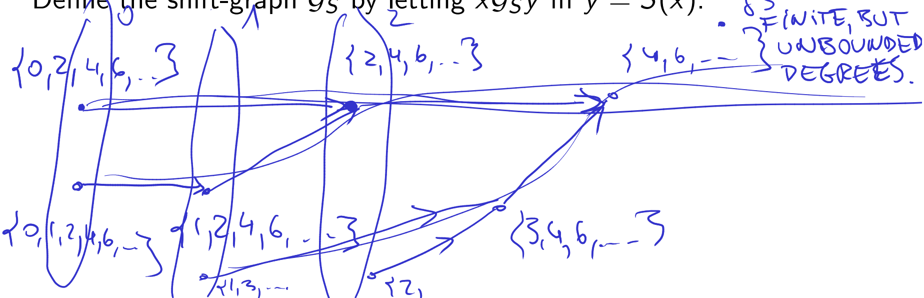
Let $S : \underset{\overset{\in}{x}}{[\mathbb{N}]^{\mathbb{N}}} \to [\mathbb{N}]^{\mathbb{N}}$ be the *shift-map*, defined by

$$S(x) = x \setminus \{\min x\}.$$

Define the shift-graph $\mathcal{G}_S$ by letting $x \mathcal{G}_S y$ iff $y = S(x)$.

- $\chi_B(\mathcal{G}_S) \leq \aleph_0$
- $\mathcal{G}_S$ ACYCLIC
- FINITE, BUT UNBOUNDED DEGREES.



The shift graph

Theorem. (Kechris-Solecki-Todorčević) $\chi_B(G_S) = \aleph_0$.

Pf: IF $c : [\mathbb{N}]^{\mathbb{N}} \to n$ IS BOREL THEN $(c^{-1}(\{i\}))_{i \leq n}$ IS A BOREL COVERING OF $[\mathbb{N}]^{\mathbb{N}}$ AND BY G-P $\exists$ $A \in [\mathbb{N}]^{\mathbb{N}}$ $\exists i$

$$[A]^{\mathbb{N}} \subseteq c^{-1}(\{i\}) \Rightarrow c(A) = c(S(A)) = i$$

↯.

KST: $\chi_B(G) > \aleph_0 \iff G_0 \leq_B G$

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Theorem. (Kechris-Solecki-Todorčević) $\chi_B(G_S) = \aleph_0$.

Question. Assume that $\mathcal{G}$ is an acyclic Borel graph with $\chi_B(\mathcal{G}) \geq \aleph_0$. Is $\mathcal{G}_S \leq_B \mathcal{G}$?

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Theorem. (Todorčević-V) There is no meaningful characterization of Borel graphs with Borel chromatic number $< n$, for each $n \in \{4, \ldots, \aleph_0\}$.

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$\chi_B(\mathcal{G}) \geq \aleph_0 \iff \nexists\ \mathcal{G}_S \to \mathcal{G}$

$\forall \quad \exists$

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$\{B \subseteq [\mathbb{N}]^{\mathbb{N}} \mid \chi_B(\mathcal{G}_S \restriction B) \leq 3\}$

Σ^1_2 - COMPLETE

In fact, the set of such graphs is Σ^1_2-complete.

$\exists$ DEF'N $\quad \exists \quad \forall$

$\nexists$ DEF'N $\quad \forall \quad \exists$

Gadgets and measures

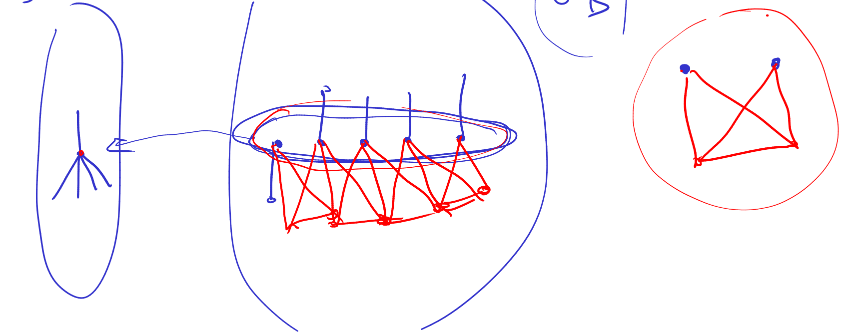
Theorem. (Grebík-V.) There is some d for which acyclic d-regular Borel graphs with Borel chromatic number ≤ 3 form a Σ^1_2-complete set.

1. UNBOUNDED $\longrightarrow$ BOUNDED (MAYBE CYCLIC)

$$\chi_B(G_S \restriction B) \leq 3 \iff \chi_B(G^*_B) \leq 3$$

$G_S \restriction B$

G^*_B



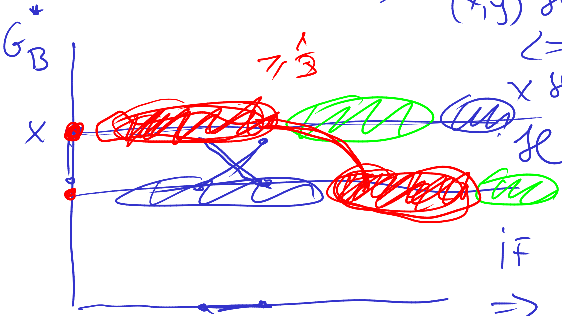
Gadgets and measures

② CYCLIC $\longrightarrow$ ~~ACYCLIC~~ ONLY EVEN CYCLES

$(x,y)\ \mathcal{H}\times G_B^*\ (x',y')$

$\iff$

$x\ \mathcal{H}\ x'$ AND $y\ G_B^*\ y'$



$\mathcal{H}\times G_B^* \leq_B \mathcal{H}$

$\leq_B G_B^*$

if $\mathcal{H}$ is ACYCLIC

$\mathcal{H}, M \Rightarrow \mathcal{H}\times G_B^*$ CONTAINS ONLY EVEN CYCLES

$\chi_B(G_B^*) \leq 3 \iff \chi_B(G_B^*\times\mathcal{H}) \leq 3$

$\Longrightarrow$ ✓

$\Longleftarrow$

Gadgets and measures

Theorem. There exists a d, an acyclic d-regular Borel graph $\mathcal{H}$ on a probability measure space (X, μ) such that for every $B \subseteq X$ Borel with $\mu(B) \geq \frac{1}{3}$ we have $\mu(N_{\mathcal{H}}(B)) > \frac{2}{3}$.

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(MARKS) $\exists\, \mathcal{G}$ 3 REGULAR ACYCLIC BOREL GRAPH WITH $\chi_B(\mathcal{G}) = 4$

Complexity on the shift

Theorem. (Todorčević-V) There is no meaningful characterization of Borel graphs with Borel chromatic number at most n, for each $n \in \{3, \ldots, \cancel{\aleph_0}\}$: such graphs form a Σ^1_2-complete set.

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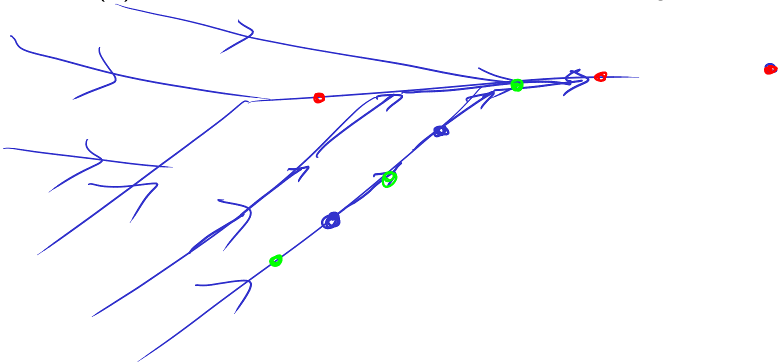
(di Prisco-Todorčević-Miller) Assume that $B \subset [\mathbb{N}]^{\mathbb{N}}$ is a Borel set, and for some $\mathcal{G}_S$-independent Borel set B', for each $x \in B$ there is n with $S^n(x) \in B'$. Call such B' and *independent hitting set*.



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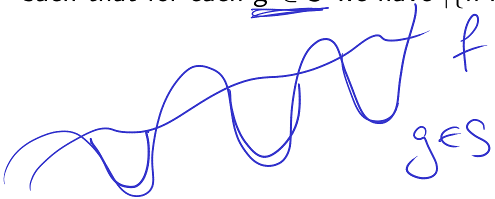
A set $S \subset [\mathbb{N}]^{\mathbb{N}}$ is called *non-dominating* if there is an $f \in [\mathbb{N}]^{\mathbb{N}}$ such that for each $g \in S$ we have $|\{n : g(n) \leq f(n)\}| = \aleph_0$.



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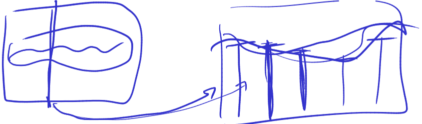

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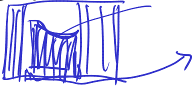
Theorem. If a Borel coloring problem is solvable on non-dominating Borel sets, and not solvable on $[\mathbb{N}]^{\mathbb{N}}$, then the Borel subgraphs of $\mathcal{G}_S$ on which it is solvable form a Σ^1_2-complete.



$\rightarrow \Sigma^1_1 \longrightarrow$

Shift and determinacy

Theorem. (B-C-G-G-R-V) Let $\mathcal{H}$ be a locally countable Borel graph. Then we have

3-REG ACYCLIC

$$\chi_{w\Delta^1_2}(\mathcal{H}) > 3 \;\Rightarrow\; \chi_B(Hom(T_3, \mathcal{H})) > 3.$$

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$\chi_B(\mathcal{G}_S \restriction B) \leq 3$

Thus,

- If $B \subset [\mathbb{N}]^{\mathbb{N}}$ is non-dominating then $\chi_B(Hom(T_3, \mathcal{G}_S \restriction B)) \leq 3$, ✓

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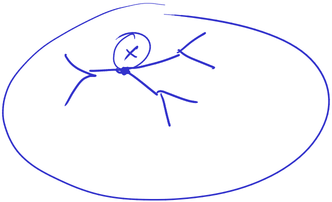
gl



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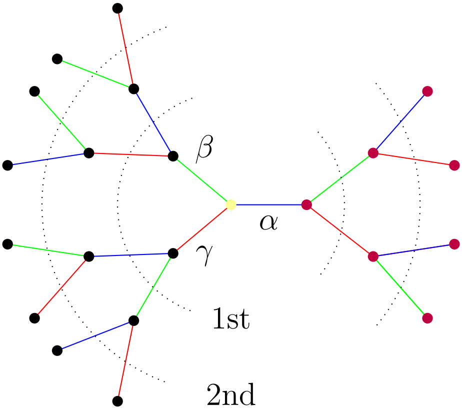
Theorem. 3-regular acyclic Borel graphs with Borel chromatic number ≤ 3 form a Σ^1_2-complete set.

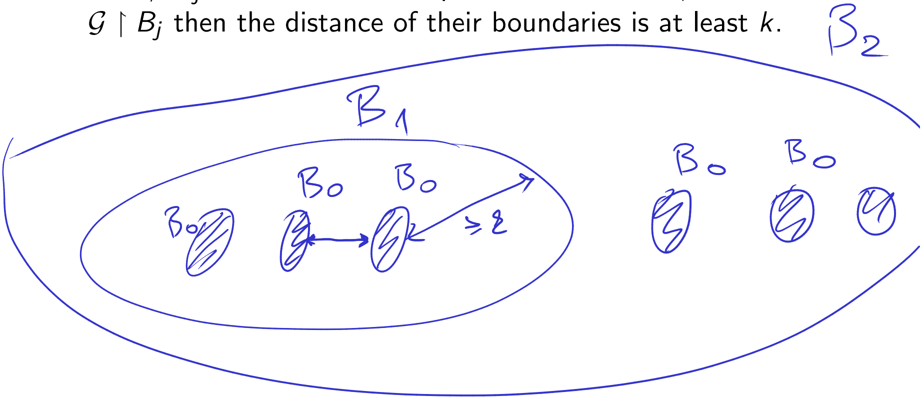
Marks' method



Toasts

Let $\mathcal{G}$ be a locally countable Borel graph and k. A *k-toast* is a sequence of Borel set $B_0 \subset B_1 \subset \cdots$ with

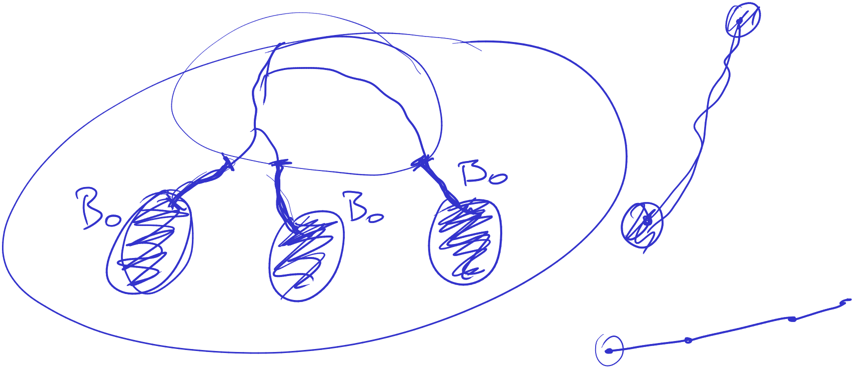
- $\bigcup_i B_i = V(G)$,
- $\mathcal{G} \restriction B_i$ has finite connected components,
- if $S_i \neq S_j$ are connected components of some $\mathcal{G} \restriction B_i$ and $\mathcal{G} \restriction B_j$ then the distance of their boundaries is at least k.



Toasts

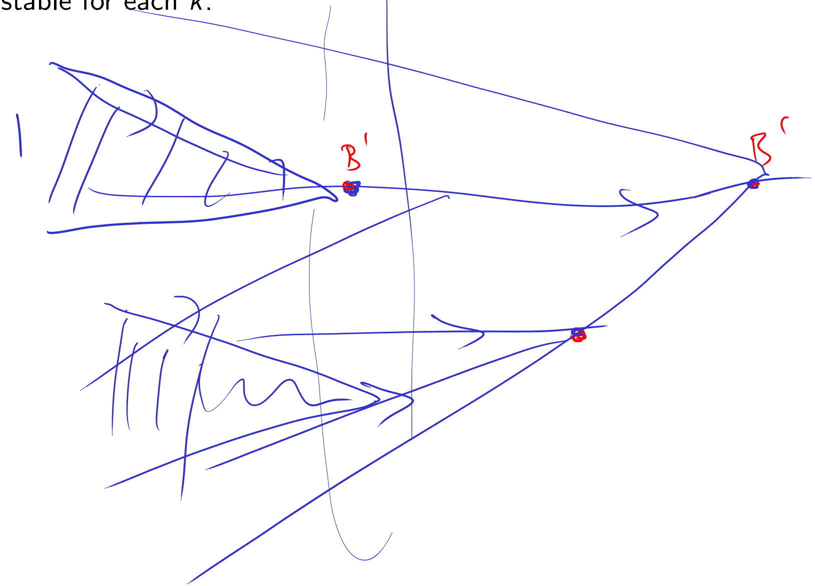
Theorem. Let l be odd. Then a k-toastable acyclic Borel graph admits a Borel homomorphism into C_l for every large enough k.

$C = 3$ $\qquad$ C_3



Toasts and non-dominating sets

Theorem. Let $B \subset [\mathbb{N}]^{\mathbb{N}}$ be non-dominating. Then $\mathcal{G}_S \restriction B$ is k-toastable for each k.



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Hell-Nesetřil

Theorem. (Hell-Nesetřil) Let H be a finite graph. Deciding whether a finite graph G admits a homomorphism into H is ~~Σ^1_2-complete.~~ NP-COMPLETE, UNLESS H IS BIPARTITE

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Combining the above theorems, we obtain a new, algebra-free strengthening of Thornton's result.

Open questions

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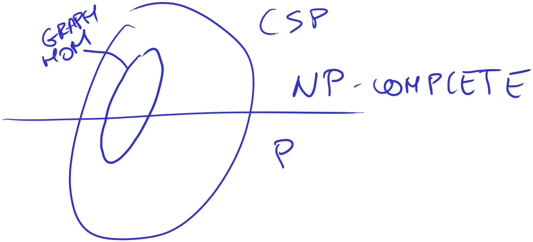
- What are the Borel CSP's that are solvable from toasts?



Thank you for your attention!